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A METHOD FOR SYSTEMIC RISK ESTIMATION BASED ON CDS INDICES

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Abstract: *The copula-GARCH approach provides a flexible and versatile method for modeling multivariate time series. In this study we focus on describing the credit risk dependence pattern between real and financial sectors as it is described by two representative iTraxx indices. Multi-stage estimation is used for parametric ARMA-GARCH-copula models. We derive critical values for the parameter estimates using asymptotic, bootstrap and copula sampling methods. The results obtained indicate a positive symmetric dependence structure with statistically significant tail dependence coefficients. Goodness-of-Fit tests indicate which model provides the best fit to data.*
Keywords*: copula, CDS, tail dependence, systemic risk.*
JEL Classification*: C15, C32, C51.*

1 INTRODUCTION

We propose a framework for systemic risk analysis based on a copula-GARCH type of model that allows for separate specification of the dependence structure from the marginal distributions. The empirical part analyses the credit risk dependence pattern, between real and financial sectors, at European level, as reflected by the iTraxx Senior Financial and Crossover CDS indices. These indices are highly representative aggregated indicators of credit risk in their respective sectors. Their most important features can be summarized as: forward looking indicators reflecting both market expectation and fundamentals with high liquidity maintained by a broad investor base and updated composition due to half-year roll-over. In addition to flexibility this set-up allows us to focus on individual risks of each sector as well as on the dependence structure that drives their joint behavior and leverages total risk.

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The literature on copula modeling is growing constantly. For a thorough introduction to copulas we refer to Joe (1997) and Nelsen (2006), the two key textbooks on dependence modeling from a statistical perspective. McNeiletal. (2005) provides a sound implementation of copula models in the context of quantitative risk management, while Embrechts et al. (2002) introduces the static representation of dependence via copulas. Patton (2006) lays down the foundation for multivariate financial time series applications of copulas and complements his research with a comprehensive empirical study in Patton (2012). Choros et al. (2010) presents the parametric and semi-parametric estimation methods for copulas on time series data. Patton (2009) gives an overall survey of copula applications to time series. Genest et al. (2009) and Remillard (2010) are two key references for Goodness-of-Fit tests for copulas.

The seminal finding in the copula modeling literature is due to Sklar (1959) who provides the formal approach to separate a joint distribution into independent margins and a copula. Accordingly, for every p-dimensional distribution F with corresponding margins F_i there exists a copula C such that:

$$F(y_1, \dots, y_p) = C(F_1(y_1), \dots, F_p(y_p)) \tag{1}$$

which is unique if all margins are continuous. Conversely:

$$C(u_1, \dots, u_p) = F\left(F_1^{-1}(u_1), \dots, F_p^{-1}(u_p)\right) \tag{2}$$

where $u_i = F_i(y_i), i = 1, \dots, p$. If F_i is continuous then the probability integral transformation$U_i = F_i(y_i)$is unique and $Unif\ (0,1)$distributed regardless of the original distribution of F_i. If $F$is p-times differentiable then the joint density is given by:

$$\begin{aligned} f(y) &= \frac{\partial^p}{\partial y_1 \partial y_2, \dots, \partial y_p} F(y) \\ &= \prod_{i=1}^{p} f_i(y_i) \frac{\partial^p}{\partial u_1 \partial u_2, \dots, \partial u_p} C\left(F_1(y_1), \dots, F_p(y_p)\right) \end{aligned} \tag{3}$$

The copula-GARCH class of models assumes certain parameters are time varying in an auto-regressive manner and their distributions are conditional on past information. In the context of this analysis we are interested in modeling the cross sectional dependence between time series data and therefore we employ an adapted version of Sklar's theorem introduced by Patton (2006). The distribution of Y_t conditional on$\mathcal{F}_{t-1} = \{Y_l : l \leq t-1\}$ is decomposed into its conditional marginsand the corresponding conditional copula in the following manner:

$$\begin{aligned} &F(y|\mathcal{F}_{t-1}) = C\left\{F_1(y_1|\mathcal{F}_{t-1}), \dots, F_p(y_p|\mathcal{F}_{t-1})\right\} \\ &\text{with} \quad Y_{it}|\mathcal{F}_{t-1} \sim F_i(\cdot\,|\mathcal{F}_{t-1}), i = 1,2, \dots, p \end{aligned} \tag{4}$$

Fitting a copula on the unconditional probability integral transform will result in an unconditional copula model for the dependence. In a time series context however, it is necessary to condition on the available past information, which first requires the specification of the margins and then the copula that joins the series cross-sectionally. If we define the probability integral transform$U_{it} = F_i(Y_{it}|\mathcal{F}_{t-1})$, then the conditional copula of $Y_t|\mathcal{F}_{t-1}$ is given by $U_t|\mathcal{F}_{t-1} \sim C(\cdot|\mathcal{F}_{t-1})$. It isimportant to note here that both the margins and the copula have to be conditional on the same data set. Fermanian and Wegkamp (2012) have shown that using different information sets for conditional margins and copulas may result in a function F that is not a valid joint distribution. In empirical studies however, it may happen that marginal distributions are only dependent on their own lags, that is they are dependent only on a fraction of $\mathcal{F}_{t-1}$but this still satisfies the restrictionthat margins and copula must use the same data set. We will use the standard ARMA-GARCH approach to model the univariate distributions by specifying the following general model for each series:

$$Y_{it} = \mu_i(Y_{t-1}) + \sigma_i(Y_{t-1})\epsilon_{it} \tag{5}$$

where, $\epsilon_{it}|\,\mathcal{F}_{t-1} \sim F_i(0,1), \forall\, t$. The functional form of F_i makes the model parametric because the copula is fitted on the conditional distribution of the probability integral transform of the standardized residuals constructed as:

$$\hat{\epsilon}_{it} = \frac{Y_{it} - \mu_i(Y_{t-1})}{\sigma_i(Y_{t-1})}, i = 1,2,\dots,p \tag{6}$$

In general financial time series have fatter than normal tails and asymmetric shape. In particular, the tail distribution of the ARMA-GARCH process is heavier than that of a normal distribution. Thus, the parametric form of F_i has to be able to accommodate thicker than normal tails and possibly an asymmetric shape. With the correct functional form the Skew-t distribution can fulfill these requirements. In this study we tested different forms of skewed distributions and choose the flexible and thoroughly studied Skew-t form of Lambert and Laurent (2001), for its suitability to the data at hand. In addition to the location and scale this distribution has two more shapeparameters: the degrees of freedom $\nu \in (2,\infty)$ controlling the thickness of the tails and the skewness parameter $\gamma \in (0,\infty)$ controlling the asymmetry.

The paper proceeds as follows: Section 2 describes the dependence measures used to analyze the degree and pattern of associativity in data. In Section 3 we present estimation methods for parametric copula-GARCH models and simulation techniques to make inference on estimated parameters. Section 4 reviews the Goodness-of-Fit tests applicable to copula models. In Section 5 we describe the data, present the actual implementation and show the results. Section 6 concludes.

2 DEPENDENCE MEASURES

The literature on multivariate analysis provides a broad array of dependence measures, see Nelsen (2006, Chapter 5) for an in-depth description. In this study we will focus on rank correlation, quantile dependence and tail dependence for their specific property of being scale invariant which is a valuable characteristic providing guidance on the type of copula to use in order to better describe the data at hand.

Linear correlation plays an essential role in finance theory; however it is important to mention that this concept only applies in the context of multivariate elliptical distributions. A detailed discussion of the shortcomings of relying on linear correlation is given by Embrechts et al. (2002) who calls them ”fallacies” and describes their effects to multivariate simulation, a technique which we will intensively use for making inference on parameter estimates. One important property of copulas, namely that the dependence structure as summarized by a copula is invariant under increasing and continuous transformations of the margins allows for the specification of the Spearman’s rank correlation directly as:

$$\begin{aligned} \rho_s &= \rho\big(F_1(Y_{1t}), F_2(Y_{2t})\big) \\ \hat{\rho}_s &= \frac{12}{T}\sum\nolimits_{t=1}^{T} U_{1t}U_{2t} - 3 \end{aligned} \tag{7}$$

which is simply the linear correlation of the probability integral transforms. Hence for continuous random variables is the linear correlation of their unique copula.$\rho_s$is constrained in $[-1,1]$with the boundaries being reached only if the variables are comonotonic or countermonotonic. A practical reason for using rank correlation is that it can be used to calibrate copulas to empirical data. Despite its appealing characteristics the rank correlation is only a scalar measure that gives an overall indication about the strength and sign of the dependence.

Quantile dependence measures the strength of the dependence between two random variables in the joint upper and lower tails of the support of their distribution. It can be estimated as:

$$\hat{\lambda}_q = \begin{cases} \dfrac{1}{qT}\displaystyle\sum_{t=1}^{T} 1\{U_{1t} \le q \mid U_{2t} \le q\}, & 0 < q \le 0.5 \\ \dfrac{1}{(1-q)T}\displaystyle\sum_{t=1}^{T} 1\{U_{1t} > q \mid U_{2t} > q\}, & 0.5 < q < 1 \end{cases} \tag{8}$$

and it provides a detailed description of the dependence at various points across the support of the two variables. By comparing the quantile dependences while moving away from the center towards the tails we gain information about the

asymmetry of the dependence structure and a clear indication about which type of copula is more appropriate to use.

According to Joe (1997) the concept of tail dependence relates to the amount of dependence in the upper or lower quadrant tail of the distribution. The motivation for looking at these coefficients is that they provide measures of extreme dependence and we describe them in terms of limiting conditional probabilities. Due to invariability to monotonic transformations, parametric tail dependence can be expressed directly in terms of copulas. For a bivariate copula the tail dependence coefficients are given by:

$$\lambda^L = \lim_{q \to 0^+} \frac{C(q,q)}{q} \; and \; \lambda^U = \lim_{q \to 1^-} \frac{1 - 2q + C(q,q)}{1 - q} \tag{9}$$

where, is said to have upper tail dependence if $\lambda^U \in (0,1]$ and lower tail dependence if $\lambda^L \in (0,1]$. If $\lambda^L = \lambda^U$then the copula is radially symmetric and if one coefficient is 0 than the copula is asymptotically independent in the respective tail. Calculation of these coefficients is straightforward, if the copula has a closed analytical form, as is the case with most elliptical and Archimedean copulas. Since we are particularly interested in making inference on the asymmetry of the dependence structure we will employ a slightly more involved method introduced by Chen et al. (2010) that allows for the computation of the copula parameter independently for each tail. Under the general censorship method of Chen et al. (2010) if $(U_1, U_2) \sim C$ then the upper tailcopula parameter θ^U conditional on $(U_1 > q, U_2 > q)$ is determined by maximizing the followinglog-likelihood:

$$\begin{aligned} \mathcal{L}(\theta^U) &= \frac{1}{T}\sum_{t=1}^{T} l_t(\theta^U) \\ l_t(\theta^U) &= \delta_{1t}\delta_{2t} \log c\,(\widetilde{U}_{1t}, \widetilde{U}_{2t}; \theta^U) \\ &+ \delta_{1t}(1 - \delta_{2t}) \log \frac{\partial C(\widetilde{U}_{1t}, \widetilde{U}_{2t}; \theta^U)}{\partial U_1} \\ &+ (1 - \delta_{1t})\delta_{2t} \log \frac{\partial C(\widetilde{U}_{1t}, \widetilde{U}_{2t}; \theta^U)}{\partial U_2} \\ &+ (1 - \delta_{1t})(1 - \delta_{2t}) \log C(\widetilde{U}_{1t}, \widetilde{U}_{2t}; \theta^U) \end{aligned} \tag{10}$$

where $\widetilde{U}_{1t} = max[U_{1t}, q]$ and $\widetilde{U}_{2t} = max[U_{2t}, q]$that is values less then q are replaces by q. Indicator functions $\delta_{1t} = 1\{U_{1t} > q\}$ and $\delta_{2t} = 1\{U_{2t} > q\}$ are used to retain in the log-likelihood function only the terms that are greater than q. Maximizing $\mathcal{L}$ requires closed forms for copula and partialdensities but these are readily available in most cases.

Inference on the estimated dependence statistics can be conducted either by using their asymptotic distribution or via simulation. We implemented a simulation algorithm based on bootstrapping to construct confidence intervals. Regardless of the computational burden the bootstrapping approach is rather convenient in comparison to the standard method of moments that relies on asymptotic properties. For the parametric case the algorithm is built around the stationary bootstrap of Politis et al. (1999)[Chapter, 3]. Asimilar parametric bootstrap can be found in Genest et al. (2006) for which the asymptotic validity has been established by Genest and Remillard (2008).

Algorithm 1 - Simulation procedure to determine confidence intervals for tail dependence statistics based on parametric marginal distributions:

(1)Estimate the model for margins and obtain the probability integral transforms (U_{1t}, U_{2t})

(2)Compute the dependence statistics $\{\tilde{\lambda}^q, \tilde{\lambda}^L \tilde{\lambda}^U\}$using the probability integral transformsfrom the actual data.

(3)Generate N samples of length T using the stationary bootstrap method to preserve the time dependence of data

(4) Estimate the model for margins on each of the N samples and obtain the probability integral transforms $\{U_{1t}^n, U_{2t}^n\}_{n=1}^N$

(5)Compute the dependence measures on each of the N samples to obtain $\{\tilde{\lambda}_n^q, \tilde{\lambda}_n^L, \tilde{\lambda}_n^U\}_{n=1}^N$

(6)Compute the $\alpha/2$ and $1-\alpha/2$ quantiles of$\{\tilde{\lambda}_n^q, \tilde{\lambda}_n^L, \tilde{\lambda}_n^U\}_{n=1}^N$intervalsto obtain $1-\alpha$ confidence intervals

(7) Compute p-values for the dependence statistics as: $p_n^i = \sum_{n=1}^N 1\{\tilde{\lambda}_n^i \geq \lambda_i\} / N,\ i \in q,L,U$

3 ESTIMATION AND INFERENCE

Estimation for copula models is usually done via Maximum Likelihood (ML). One may attempt to estimate margins and copula in one single optimization, however splitting the modeling into two steps can yield more insight and allow for more detailed analysis of the different model components. When margins are estimated parametrically the model is called parametric and inference about the copula amounts to what has been termed the inference-functions for margins (IFM) approach of Joe (1997). The properties of estimates derived using the single stage versus the IFM method, have both been studied in great theoretical detail in the

literature. In practice, estimation requires numerical optimization of the log-likelihood which in terms requires the derivation of copula density.

When using parametric models for the conditional distributions and the copula the log-likelihood is completely specified and the parameters can be estimated as:

$$\begin{gathered}\hat{\theta}_{ML} = \arg\max \log \mathcal{L}(\theta) \\ \log \mathcal{L}(\theta) = \sum_{t=1}^{T}\sum_{i=1}^{p} \log f_i(Y_{it};\theta_1) \\ + \sum_{t=1}^{T} \log c\big(F_1(Y_{1t};\theta_1), \dots, F_p\big(Y_{pt};\theta_1\big);\theta_2\big)\end{gathered} \tag{11}$$

where $\theta = [\theta_1, \theta_2]$ is the parameter vector for the full model.

In the following we will assume that the regularity conditions of Joe (1997) for parameter asymptotic distribution hold for the multivariate model as well as for all margins. Under these regularity conditions the maximum likelihood estimator exists, is consistent and asymptotically efficient and it has the property of being asymptotically normal:

$$\sqrt{T}\big(\hat{\theta}_{ML} - \theta\big) \to N(0, V_{ML}) \; for \; T \to \infty \tag{12}$$

where $V_{ML}$is the covariance matrix of $\hat{\theta}_{ML}$ which can be estimated by the inverse of the Hessianmatrix of the log-likelihood function using standard methods:

$$\begin{gathered}\hat{V}_{ML} = \hat{A}^{-1}\hat{B}\hat{A}^{-1} \\ \hat{A} = \frac{1}{T}\sum\nolimits_{t=1}^{T} \widehat{H}_t \; and \; \hat{B} = \frac{1}{T}\sum\nolimits_{t=1}^{T} \hat{s}_t \hat{s}_t' \\ \hat{s}_t = \frac{\partial}{\partial\theta} \log f_t\big(Y_t;\hat{\theta}_{ML}\big) \\ \widehat{H} = \frac{\partial^2}{\partial\theta\partial\theta'} \log f_t\left(Y_t;\hat{\theta}_{ML}\right)\end{gathered} \tag{13}$$

The canonical representation for the multivariate density function in (3) permits us to say that, in general, a statistical modeling problem for copulas could be decomposed into two steps: identification of the marginal distributions and specification of the appropriate copula function. Thus, IFM parameters are estimated as in:

$$\hat{\theta}_1 = \arg\max \sum_{t=1}^{T}\sum_{i=1}^{p} \log f_i(Y_{it};\theta_1) \tag{14}$$

$$\hat{\theta}_2 = \arg\max \sum_{t=1}^{T} \log c\left(F_1(Y_{1t};\theta_1), \dots, F_p\left(Y_{pt};\theta_1\right);\theta_2\right)$$

and $\hat{\theta}_{IFM} = [\theta_1, \theta_2]$ is the parameter vector estimated in stages. A critical point here is that asymptotic normality does not hold under model misspecification. This implies that the copula likelihood being maximized is not the true likelihood because it is influenced by parameter estimation error in the margins. The naive approach would be to just use the inverse Hessian of the copula likelihood as an estimate of the covariance matrix of the copula parameters but the result should be used with caution as it fails to account for the additional source of error. In order to incorporate the estimation error that propagates from the margins to the copula the following correct estimator should be used for the covariance matrix:

$$\begin{gathered}
\hat{V}_{ML} = \hat{A}^{-1}\hat{B}(\hat{A}^{-1})' \\
\hat{A} = \frac{1}{T}\sum_{t=1}^{T} \hat{H}_t \ \text{and}\ \hat{B} = \frac{1}{T}\sum_{t=1}^{T} \hat{s}_t\,\hat{s}_t' \\
\hat{s}_t = \left[\hat{s}_{1t}, \dots, \hat{s}_{pt}, \hat{s}_{ct}\right] \\
\hat{s}_{it} = \frac{\partial}{\partial\theta_{1i}} \log f_t\left(Y_t;\hat{\theta}_{1i}\right), i = 1,2,\dots,p \\
\hat{s}_{ct} = \frac{\partial}{\partial\theta_2} \log c\ \left(F_1\left(Y_{it};\hat{\theta}_{1,1}\right), \dots, F_p\left(Y_{pt};\hat{\theta}_{1,p}\right);\hat{\theta}_2\right)
\end{gathered} \tag{15}$$

Hessian has to be adjusted to account for marginal parameter estimation error. Asymptotic theory represents the first choice to estimate the parameter covariance matrix. The alternative is to compute standard errors by simulation in either of two ways: by stationary bootstrap or by simulation from copula.

Algorithm 2 - Simulation procedure to compute parametric ML standard errors

(1) Estimate the parametric model on actual data using the IFM

(2) Simulate via stationary bootstrap

(2.1) Generate N samples of length T using the stationary bootstrap method to preserve the time dependence of actual data

(3) Simulate via copula sampling

(3.1) Simulate N samples of length T by sampling from the copula

(3.2) Generate the standardized residuals by transforming the data using the inverse of the marginal conditional distribution

(3.3) Use the standardized residuals to recursively generate the simulated data using the fitted ARMA-GARCH process

(4) Estimate the same model on each of the N samples

(5) Use the distribution of $\{\hat{\theta}_n\}_{i=1}^{N}$ to obtain the standard errors

It is important to note that both the bootstrapping and copula simulation preserve the cross-sectional dependence of data. Time dependence is implicitly insured by the stationary bootstrap while in the copula sampling case it is induced by the recursive nature of the data generating process at Step (3.3).

4 GOODNESS-OF-FIT TESTS

Goodness-of-Fit (GoF) tests have become standard tools of summarizing model performance. We focused on in-sample Kolmogorov-Smirnov (KS) and Cramer-von Mises (CvM) test to assess the parametric distribution of the standardized residuals and copula performance. These are two of the most powerful test available. Our choice is further motivated by their suitability for copula inference and a broad scientific literature describing their theoretical characteristics. In the univariate case GoF tests provide the proper means to test if a set of observations comes from a completely specified continuous distribution. We use GoF tests to assess if the distribution of the AR-GARCH residuals is well-specified, or more specifically, to test whether the estimated probability integral transformations of the residuals come from a distribution that is statistically different from the cumulative distribution function of $Unif\ (0,1)$. This assessment is carried out by using the KS and CvM tests, as defined below:

$$KS_i = \max\left|\widehat{U}_{it} - \frac{t}{T}\right|$$
$$CvM_i = \sum_{t=1}^{T}\left(\widehat{U}_{it} - \frac{t}{T}\right)^2 \tag{16}$$

where $\widehat{U}_{it} = Skew - t(\hat{\epsilon}_{it}; \hat{\nu}_i, \hat{\gamma}_i)$is the t^{th} order statistic of $\{\widehat{U}_{it}\}_{t=1}^{T}$with $i = 1,2,\dots,p$. Unfortunately, when certain parameters for the distribution have to be estimated the KS and CvM testsdo not follow known distributions and their critical values have to be determined by simulation.In the multivariate case these tests share the same idea by using the empirical copula as a reference. The empirical copula converges uniformly to the underlying true copula and Nelsen (2006) notes that this concept permits us to define the sample version of many dependence measures as well as non-parametric tests for independence. Genest et al. (2009)

uses a simulation study to compare a range of copula GoF tests and concludes that the most powerful is the CvM test applied to the Rosenblatt transforms of the original data. This finding is also supported by Berg(2009) who considers a broader range of tests. Following their findings we conduct inference on model specification directly on the Rosenblatt transforms of the data generated by the fitted copula model. Broadly speaking, Rosenblatt transform represents the multivariate version of the probability integral transformation mapping the original data into a hypercube of mutually independent variables. In the bivariate case if $Z_{1t} = U_{1t}, \forall t$ and $Z_{2t} = C_{(2|1t)}(U_{2t}|U_{1t};\theta)$ where$C_{(2|1t)}$is thedistribution of U_{2t} conditional on U_{1t} then, according to Rosenblatt (1952), $Z_{1t}$and Z_{2t} are iid and $Unif\,(0,1)$ distributed. This approach can be extended to multiple dimensions by considering:

$$Z_{it} = \frac{\partial^{i-1} C(U_{1t}, \dots, U_{it}, 1, \dots 1)}{\partial u_1, \dots, \partial u_{i-1}} \Big/ \frac{\partial^{i-1} C\big(U_{1t}, \dots, U_{i-1,t}, 1, \dots 1\big)}{\partial u_1, \dots, \partial u_{i-1}}, i = 2, \dots, p \quad (17)$$

The advantage of this transformation is that $Z_t \sim C_{indep}$ if the conditional copula is correctly specified. Assuming that the empirical copula, as defined in (18), is the closest estimate to the true conditional copula we can use the KS_R and CvM_R, as defined in (20) and (21), to test if the independence copula defined in (19) is statistically different from the empirical copula estimated on the Rosenblatt transforms.

$$\hat{C}_R(z) = \frac{1}{T}\sum_{t=1}^{T}\prod_{i=1}^{p} 1\{Z_{it} \le z_i\} \quad (18)$$

$$C_R\big(Z_t;\hat{\theta}\big) = \prod_{i=1}^{p} Z_{it} \quad (19)$$

$$KS_R = \max\Big|C_R\big(Z_t;\hat{\theta}\big) - \hat{C}_R(Z_t)\Big| \quad (20)$$

$$CvM_R = \sum_{t=1}^{T}\big\{C_R\big(Z_t;\hat{\theta}\big) - \hat{C}_R(Z_t)\big\}^2 \quad (21)$$

In the absence of parameter estimation error GoF test statistics follow some known distribution (i.e.$\chi 2$). However, known parametric distributions are not applicable here because $KS_R$and CvM_R depend on $\hat{\theta}$, that is they are affected by estimation error both in marginal distributions and copula. This is the reason why simulation represents the most suitable method for determining p-values for KS_R and CvM_R tests. In the parametric case the simulation procedure is straight forward as it involves sampling and estimation of both marginal distributions and copula. Algorithm 3 describes the simulation procedure to generate viable p-values for KS_R and CvM_Rtests for parametric models.

Algorithm 3 - Simulation based procedure to determine copula GoF p-values

(1) Estimate the full model (margins and copula) on the actual data to obtain the vector $\hat{\theta}$ of parameter estimates

(2) Compute the GoF test statistics $G = \{KS;\ CvM\}$ on the original data set

(3) Simulate N samples of length T from the full model using the estimated parameter $\hat{\theta}$

(4) Estimate the full model on each of the N simulated samples to obtain $\hat{\theta}^{(n)}$

(5) Compute the GoF test statistics for each of the samples to obtain $G^{(n)} = \left\{KS^{(n)}, CvM^{(n)}\right\}_{n=1}^{N}$

(6) Compute the simulation based p-values as $p_{iN} = \sum_{n=1}^{N} 1\left\{G_i^{(n)} \geq G_i\right\}/N\,, i = 1,2$

5 IMPLEMENTATION AND RESULTS

Our goal is to describe the dependence between the iTraxx Europe Senior Financial and Crossover CDS indices as a measure of systemic risk between the real and financial sectors at European level. The study is divided in two phases. In the fist phase, we analyze the dependence pattern between the two series and make inference on their joint tail behavior in order to get an insight on what class of copula would be able to capture the dependency structure. In a second phase, we fit a copula-GARCH model and test for the statistical significance of the estimated parameters. One advantage of the copula-GARCH approach is the possibility to specify the model in stages. The marginal distributions are specified by an ARMA-GARCH model for each univariate time series and then a copula is estimated on the probability integral transforms of the standardized residuals. The result is a valid 2-dimensional joint distribution that is easier to estimate and interpret.

The Markit iTraxx Europe Senior Financial index comprises 25 equally weighted CDSs on investment grade European financial entities whereas the Markit iTraxx Europe Crossover index comprises 50 equally weighted CDSs on the most liquid sub-investment grade European corporate entities. The iTraxx indices are rules-based meaning that the selection methodology ensures they are replicable and represent the most liquid, traded part of the market. Composition of each index is determined by a liquidity poll and certain other criteria set up by the index rules. The constituents are changed every six months, on March 20 and September 20, in a process known as ”rolling” the index. Every time the index in rolled a series is created and for the time until the next roll it is called the on-the-run series. Investors can express their bullish or bearish sentiments related to the credit risk as an asset

class which makes the iTraxx indices very efficient as forward looking indicators. Given their liquidity and actualized composition we believe these indices are representative indicators of credit risk for their respective sectors. When analyzed together they reveal the interdependencies between the credit risk of financial and real sectors and therefore create a good reflection of systemic risk at European level. For the sake of brevity we will refer to these indices as SenFin and CROSS.

We proceed in analyzing the log-returns of the end-of-day quotations for the two indices with 5 years maturity. In order to construct a consistent time series we stitched together the log- returns of Series 5 to Series 19 creating a total of 1880 observations spanning the time period from 2006-03-21 to 2013-06-17. Each index series contributes to the final time series of returns only on the period while it is on-the-run (roughly 6 months). Every time a new series is created not only that it may have different constituents but its actual maturity is 6 months longer then the previously on-the-run series. This difference in maturities alone translates into a spread of about 10-15 bp between the old and new series. In order to avoid breaks the actual switch from one series to the other was made in the week following the roll-over on the day when the difference in returns calculated for the two series was the lowest. The return series time span is long enough to include both pre and post crisis market conditions and therefore it is representative of the long run connection between the credit risk of financial and real sectors. All data was retrieved from the Bloomberg Database.

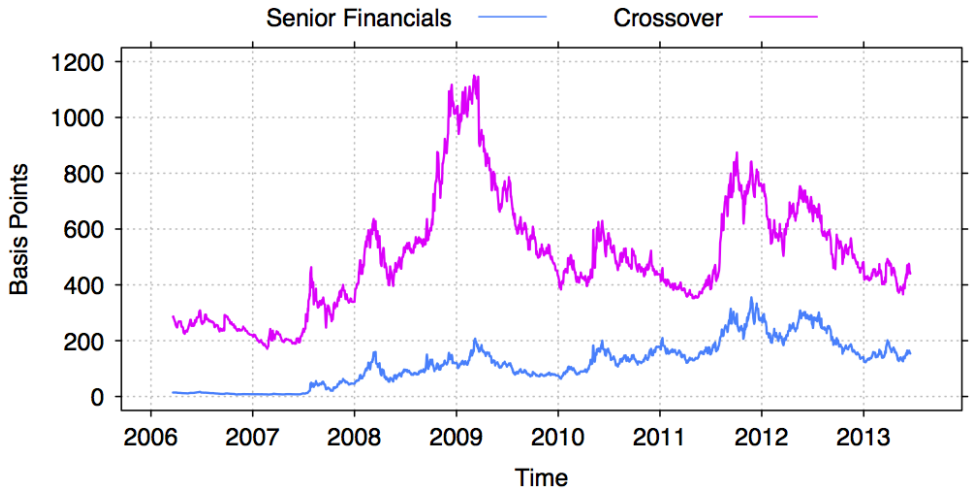


Figure 1: iTraxx Senior Financial and Crossover index levels

Figure 1 depicts the levels of the two indices during the analyzed period. There is clearly more variability in the CROSS index explained by its composition of sub-investment grade CDSs that react more vigorously to changing economic

conditions. Although not as volatile the SenFin index shows a steady upward trend indicating increasing risk in the financial sector. It is interesting to see that selling protection on SenFin was almost worthless up to September 2007 giving the false impression that financial institutions could not default. Another evident aspect form this chart is that the two indices have become more correlated from 2010 onwards. The support received from the European authorities benefited the financial institutions and kept SenFin index relatively stable hovering around 200 bp for the past 2 years. CROSS index followed the same path but with larger swings. An interpretation of this fact would be that systemic risk is not driven by credit risk of financial institutions but rather their inability to finance the real sector.

Table 1. Summary statistics for the returns of the two indices

	Financial	**Crossover**
Mean	0.1083	-0.0348
Std dev	4.7961	2.8720
Skewness	0.3146	0.0488
Kurtosis	10.7897	3.1590
Corr Pearson	0.7804	
Corr Spearman	0.7907	

The summary statistics presented in Table 1 indicate the returns on the two indices have fat tails and possibly skewed distributions. Both linear and rank correlation coefficients give a first indication that risks in the two sectors are dependent. Understanding the dependence pattern is important when choosing the copula to model this dependence. The scatter plot in Figure 2 gives a visual interpretation to the correlation coefficients in Table 1 and clearly reveals an elliptical shape with positive dependence between the returns of the two indices. This pattern is well-suited for the classes of elliptical as well as Archimedean copulas. The visual inspection of the dependence enables us to make several more assumptions that will be statistically tested later on: there is asymmetric dependence with returns more concentrated in the lower tail, there appears to be more mass in the upper tail and extreme returns tend to be correlated.

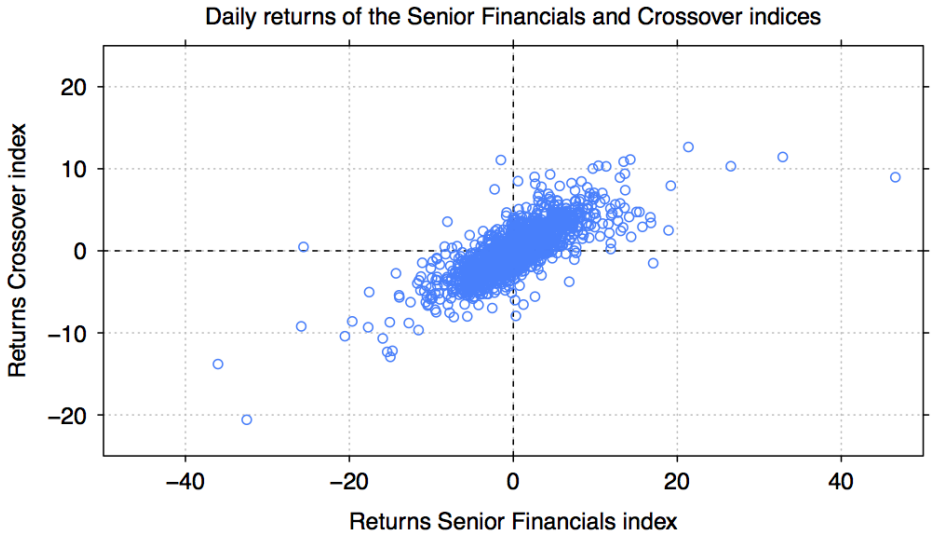


Figure 2. Scatter plot with isometric scales of daily returns of the two indices

Applying an ARMA-GARCH process to the return series of the two indices removes the temporal correlation but preserves the cross-sectional dependence. As most asset returns, our series are not stationary and therefore we followed the standard mean-variance model building approach to make our series temporal independent. The order for the mean equation was determined by comparing the BIC of ARMA models of orders up to 5. An AR(1) process was optimal for both series and this decision has also been confirmed by the partial autocorrelation function of the squared log-returns. Then, we used the squared residuals from the mean equation to test for ARCH effects. Applying the Ljung-Box test on the first 12 squared log-returns revealed p-values very close to zero which gives strong indication of rejecting the null and a motivation to introduce a conditional variance equation. In addition to auto-regressive models we tested for cross-correlation by regressing one series on up to 5 lags of the other series. The coefficients came out not significant at 5% level. This finding together with the shape of the scatter plot in Figure 2 is a strong indication that the two series are not linearly correlated but dependent. Empirical studies have shown that asset returns have fatter tails than the normal distribution can capture. Besides fat tails, empirical distributions of asset returns may also be skewed. To handle these characteristics we used a skewed version of the Student-t distribution. We adopted the specific form of Lambert and Laurent (2001) for its suitability to ARCH processes. Taking all the above into consideration, we implemented this form of mean-variance modeling:

$$r_{i,t} = \phi_0 + \phi_1 r_{i,t-1} + e_{i,t}$$
$$e_{i,t} = h_{i,t}\epsilon_{i,t}, \epsilon_{it} \sim Skew - t(\nu, \gamma) \quad (22)$$
$$h_{it} = \omega_{i,t} + \alpha e_{i,t-1}^2 + \beta h_{i,t-1} + \delta e_{i,t-1}^2 \mathbf{1}\{e_{i,t-1} \leq 0\}$$

where $r_{i,t}$ are the log-returns of the two indices and $\epsilon_{it}$is computed according to (6). Despite the fact that we specified the model in steps estimation was performed in one single optimization using pseudo Maximum Likelihood. Empirical research has also found that returns display alternating periods of high and low volatility with asymmetric impact of innovations. This is why we introduced the asymmetry parameter δ, to capture this kind of leverage effects, however it came out to be insignificant and was dropped from the model. Again, we used BIC to chose among the various AR-GARCH models tested and the results revealed that an AR(1)-GARCH(1,1)-Skew-t model was optimal for both series. Table 2 presents the results of the conditional mean and variance model with standard errors based on Hessian.

Table 2. Parameter estimates for AR-GARCH processes with Skew-t residuals

	Financial		**Crossover**	
	Estimate	**Std.error**	**Estimate**	**Std.error**
ϕ_0	0.1009	0.0581	−0.1596	0.0542
ϕ_1	0.1225	0.0239	0.1137	0.0241
ω	0.2132	0.0836	0.2054	0.0670
α	0.1511	0.0227	0.1184	0.0186
β	0.8562	0.0185	0.8613	0.0204
γ	1.0244	0.0323	1.0109	0.0322
ν	5.9909	0.8274	8.9832	1.6732

A positive mean confirmed by a positive conditional mean for the SenFin returns provides a clear indication that the European financial sector credit risk has been steadily increasing over the analyzed period. On the other hand the conditional mean for the CROSS index indicates an alleviation of credit risks in the real sector. While highly statistically significant γ indicates a very low degree of skewness for both SenFin and CROSS conditional distributions, meaning that both high and low risk situations are equally likely to occur. ν denotes fat tails and its values are consistent with the range of the returns depicted in the isometric scatter plot of Figure 2.The choice for the parametric distribution of the standardized residuals is of crucial importance because it generates the underlying data on which the parametric model is specified. The longer right tails for both SenFin and CROSS indices are consistent with the sign of the skewness parameter in Table 1 and the γ parameter.

As it was pointed out in Section 4 the KS and CvM test statistics follow a known distribution only in the absence of parameter estimation error. In the context of this analysis the conditional distributions of the residuals depend on the parameter estimates of the ARMA-GARCH processes and therefore we resorted to a simple simulation procedure, based on Genest and Remillard (2008), to determine reliable p-values: (1) simulate a sample of length T for each return series using the model with the estimated parameters, (2) re-estimate the model on the resulted sample, (3) compute the test statistics $G^{(n)} = \{KS;\ CvM\}$ on the probability integral transforms of the sample, (4)repeat steps (1)-(3) N times and (5) compute the p-values as:$p - value = \sum_{n=1}^{N} 1\{G^{(n)} \geq G\}/N$. Table 3 presents the results of the GoF tests in (16) applied to the standardized residuals of the two indices. Under the null the test assumes that the distribution is well specified. Analyzing the p-values returned after performing $N = 10^4$ simulations we fail to reject the null hypothesis and therefore we have a strong indication that the Skew-t model is well-specified.

Table 3. p-values for GoF tests for the Skew-t distribution fitted to the residuals of AR-GARCH

	Financial		**Crossover**	
	Test	***p*-value**	**Test**	***p*-value**
KS	0.0265	0.1640	0.0267	0.1120
CvM	0.2487	0.2430	0.1583	0.1870

Our reason for looking at rank correlation is because it provides information on the sign of the dependence between the two indices and gives us a valuable indication on which types of copulas to use. The rank correlation of standardized residuals, calculated according to (7), is 0.7779. Using a simple *iid* bootstrap we computed a 90% confidence interval (CI) of $[0.7597, 0.7952]$. Unlike elliptical copulas, Archimedean copulas can only capture positive dependence and having a statistically significant positive rank correlation justifies the application of these copulas.Of course the parametric copula chosen for the tail dependence must be able to capture tail dependence. Even more so, the copula must provide non-zero tail dependence in the specific tail on which is to be used. The Gumbel copula is also an extreme value copula and according to Caperaa et al. (2000), Archimedean copulas that belong to a domain of attraction are necessarily in the domain of attraction of the Gumbel copula. Hence, in an Archimedean framework, the Gumbel copula seems to be a natural choice regarding tail dependence estimation.

In order to better describe the tail dependence in our data set we have used 2 approaches: the Gumbel copula as representative of the Archimedean class and for the reasons outline above and the Student-t copula as representative of the elliptical

class. The tail dependence coefficients are computed according to (9) parameterized with the Gumbel or Student-t copulas. The parameters for each copula have been computed independently for each tail by maximizing (10). The estimated tail copula parameters are $\theta^L = 1.5469$ and $\theta^U = 1.3557$ for the Gumbel copula and $\theta^L = \{0.6590, 5.49\}$ and $\theta^U = \{0.5475, 5.53\}$ for the Student-t copula. Having a separate parameter foreach tail assures that the parametric copula is calibrated specifically to capture the dependence in that particular tail. The Gumbel copula can only capture upper tail dependency therefore we had to rotate the data to model lower tail dependency. Point-wise CI are computed using the simulation procedures described in Algorithm1. Table 4 presents the tail dependence coefficients together with 90% CI and p-values.

Table 4. Estimates of tail dependence

		Gumbel		**Student-t**	
Lower tail dependence	λ^L	0.4346		0.2885	
	90% CI	0.3368	0.5514	0.2317	0.4151
Upper tail dependence	λ^U	0.3325		0.2123	
	90% CI	0.2151	0.4298	0.1793	0.3537
$p-value\ H_0: \lambda^L = \lambda^U$		0.4414		0.3845	

A word of caution regarding the threshold level q. The threshold q has to be chosen in advance and this introduces a trade-off between variance and bias in the estimator. The closer q is to the boundary of the support the smaller the bias but the larger the variance. In practice, however, this cut-off points are somehow chosen arbitrarily. Our choice is $\{0.025, 0.975\}$.The first conclusion stemming from the results in Table 4 is that there is no asymmetricdependence in the tails. p-values indicate that we fail to reject the null on both cases. The second conclusion is that all tail dependence coefficients are statistically significant. None of the confidence intervals include zero. From a financial point of view these two conclusions provide strong evidence supporting systemic risk. This means that, at the limit, the financial and real sectors credit risks are linked regardless of the direction. Extreme credit risk events, such an economic crisis, cannot happen independently in one sector without affecting the other.

Quantile dependence parameters provide a more comprehensive description of the dependence at various quantiles. We have calculated the quantile dependence coefficients using (8) at quantiles set up at intervals of 0.025 in $[0.025, 0.975]$. Figure 3 presents the results with the gray shaded area representing the CI for the quantile dependence coefficients and the pink shaded area denoting the point-wise CI of tail dependence. CI have been calculated using Algorithm 1. The right side plot present the difference in quantile and tail dependence

coefficients with their respective CI and give a wonderful visual interpretation of the symmetric dependence. It is clear that the CI includes the zero line which indicates that we cannot reject the null $H_0 : \lambda_q = \lambda_{1-q}, \forall\, q$.

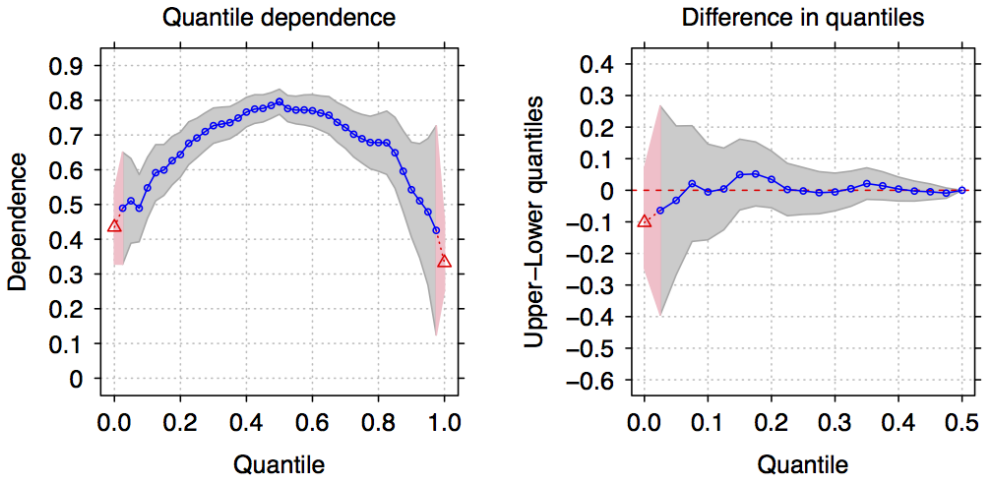


Figure 3. Quantile dependence coefficients and confidence intervals

In order to model the full dependence structure in our data set we use 2 elliptical and 2 Archimedean copulas as follows: 1) The Normal copula has become the standard tool in dependence modeling and despite its inability to capture tail dependence we use it as a benchmark. 2) The elliptical shape of the scatter plot of returns together with the findings about tail dependence lead us to believe that the Student-t copula may be the most appropriate to describe the dependence between the two return series. 3) Even though the upper and lower tail coefficients are not statistically different the quantile dependence computed for lower quantiles are higher than the ones computed for upper quantiles. This is consistent with the data being denser in the lower half of the support. In order to accommodate this empirical pattern we will use the Gumbel copula which has asymmetric contour diagrams with stronger upper tail dependence and shows more variability and mass in the negative tail. Because the characteristics of the Gumbel copula are exactly the opposite to what our data exhibits we used the rotated form.4) Clayton copula has more mass in the upper tail while creating lower tail dependence - which is consistent with larger lower quantile dependence coefficients.

In the multi-stage copula-GARCH framework estimation for copulas is straightforward because the optimization is performed over one or two parameters. One could envisage using the two-stage method to decide on the most appropriate copula family and then estimate all parameters (marginal and copula) in a final fully parametric round of estimation. Given the large sample of our data set we did

not proceed with this approach and relied on multi-stage IFM.Tables 5 presents the estimation results for parametric models using the methods described in Section 3. The first column presents the copula log-likelihood while the second presents the estimated parameters calculated according to (14). The third column presents the Kendall's τ coefficient equivalent to the copula parameters. The Kendall's τexpresses the strength of the dependence in the same units and therefore represents a feasible method of assessing the degree to which the copula is able to capture the dependence. The last four columns present the standard errors of parameter estimates calculated according to the following four methods: 1) the naive method, that is directly from the Hessian of the copula log-likelihood function, 2) the correct multi-stage ML methods described in (15), 3) simulation by stationary and *iid* bootstrap and 4) simulation by copula sampling. The simulations are performed for $N = 10^4$repetitions according to the procedure described in Algorithms 2.According to our *a priori* expectations, based on tail dependence analysis, the Student-*t* copula performed best as indicated by the largest log-likelihood value. This finding has at least two explanations. First, this copula's characteristics are closest to the pattern exhibited by our data set and in particular it is the only copula that has both lower and upper tail dependency. Second, it is the only copula with two parameters benefiting from the flexibility of a parameter that controls exactly the thickness in the (multivariate) tails.

Table 5. Copula parameter estimates and standard errors

		Parameters		**Standard errors**			
	$\mathcal{L}$	θ	τ	**Naive**	**MSML**	**Bootsrap**	**Simulation**
Normal	899.7	0.7848	0.5744	0.0071	0.0193	0.0184	0.0087
Student-t	948.9	0.7951	0.5851	0.0082	0.0152	0.0182	0.0098
		5.49		0.0261	0.1221	0.0298	0.0332
Rotated Gumbel	889.1	2.2773	0.5608	0.0433	0.1044	0.1037	0.0613
Clayton	727.8	1.7982	0.4733	0.0604	0.1737	0.1441	0.1358

The Normal copula provides the second best fit. This is somehow surprising because this copula cannot capture tail dependence. This is an indication that tail events have a low weight in setting the strength of the dependence. The rotated Gumbel copula performed relatively poor, below the benchmark set by the Normal. This stems from the fact that our data set has both lower and upper tail dependency and this copula can only properly describe one tail at a time. In this context the Gumbel copula is more suitable to describe tail dependence. Kendall's τ values are almost the same with the exception of Clayton copula which has a sensibly lower value. This finding together with the lowest log-likelihood values renders the Clayton copula inappropriate for this data set. Naive standard errors are

significantly lower than the correct multi-stage ML errors giving a false impression of consistency. This is because the naive method ignores the parameters estimation error in the marginal distributions. The results obtained are encouraging because they follow precisely the theoretical guidelines. The bootstrap and copula sampling methods returned standard errors relatively close to the correct asymptotic multi-stage ML. The conclusion is that the easy to implement bootstrapping method is reliable and can be used instead of the cumbersome MSML. However, depending on the implementation and computing power it can be pretty slow.

GoF tests may be considered too strict as they compare relative to the true model. Financial modeling in general and this study in particular relies on estimation and none of the models are expected to be correctly specified. Following this reasoning, the purpose of the GoF test in this study is to determine whether the copula model is different from the true unknown copula and not necessarily for comparison among fitted copulas. The log-likelihood values in Table 5 provide a more appropriate means of comparison between models. Considering the implementation in Genest et al. (2009) we assessed the 4 copula models using the GoF tests presented in (20) and (21) applied to the Rosenblatt transforms of the standardized residuals. Table 6 presents the p-values for the KS_R and CvM_R. Low p-values are evidence against the null that the tested copula is the true unknown copula. Student-t copula is the only one that passes the GoF tests, a result which comes in support of our findings that this is also the best copula, as indicated by the log-likelihood.

Table 5. p-values for GoF tests for selected copula models

	Normal	**Student-t**	**Gumbel**	**Clayton**
KS	0.02	0.13	0.38	0.09
CvM	0.02	0.17	0.16	0.11

6 CONCLUSIONS

We proposed a multi-stage copula-GARCH model to capture the credit risk dependence between real and financial sectors at European level. Our data set is comprised of iTraxx Senior Financial and Crossover CDS indices, highly representative for credit risk dynamics in their respective sectors. In the first part of the study we analyzed the dependence pattern and concluded that the data set has a symmetric dependence structure with statistically significant tail dependence coefficients. These findings are consistent with the existence of systemic risk. In the second part we described the entire dependence pattern by fitting four types of copulas and derived standard errors for the parameter estimates using asymptotic

theory, bootstrapping, and copula sampling. The results indicate the Student-*t* copula as the most appropriate to describe the dependence structure. Our standard error estimates follow precisely the theoretical guidelines. GoF test also indicate that the Student-*t* copula performed as accurately as possible relative to the true copula.

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